Average Strategy Fictitious Play with Application to Road Pricing

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Abstract— This paper presents a variant of standard fictitious play called *average strategy fictitious play* (ASFP) for large-scale repeated congestion games, where only a weighted running average of all other players' actions is assumed to be available to each player. It reduces the burden of both information gathering and information processing for each player. Compared to joint strategy fictitious play (JSFP) studied in the literature, the updating process of utility functions for each player is avoided. We prove that there exists at least one pure strategy Nash equilibrium for the congestion game under investigation, and the players' actions generated by ASFP with inertia (players' reluctance to change their previous actions) converge to a Nash equilibrium almost surely. The results are applied in road pricing design to achieve socially beneficial trip timing. Simulation results are provided based on the real traffic data for the Singapore case study.

I. INTRODUCTION

Game theory deals with strategic interactions among multiple players, where each player tries to maximize his or her own utility [1], [2]. It is applied in a broad array of areas including economics, social science, engineering, etc. For any non-trivial game, the utility of each player depends on choices or actions of at least one other player and generally of all the players. Nash equilibrium, a fundamental concept in the realm of noncooperative game theory, is defined as the action profile of all players where none of the players can improve his or her utility by a unilateral move. It essentially characterizes the user optimal situation, where the utility function of every myopic player reaches a local optimum.

In this paper, we consider repeated games where in each stage, the players are allowed to choose their actions based on the available information. Generally, players need to learn the underlying structure of the game by observing the decisions made by other players, especially when players have only limited or even no knowledge of their opponents' utilities. Fictitious play and its variants are classical learning processes studied extensively in the literature. In a standard fictitious play, each player computes the empirical frequencies of the observed decisions and assumes that all other

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players make decisions independently according to those empirical frequencies [3]. Several counterexamples show that fictitious play needs not converge [4], [5]. However, it has been proved by using a Lyapunov stability approach that convergence is possible under some non-singularity conditions if we have either a zero-sum game, an identical interest game, or a two-player/two-move game [6]. It is also known that the empirical frequencies generated by fictitious play of a potential game converge to a mixed strategy Nash equilibrium [7]. One obvious shortcoming of fictitious play is that when the number of players is large, it is computationally infeasible to obtain the best response for each player, since the best response of a player depends on a mapping over a joint action space of other players. See Section II for more details.

As a variant of fictitious play, a joint strategy fictitious play (JSFP) alleviates the informational and computational burden of standard fictitious play by introducing a utility updating process for each player [3], [8]. Different from standard fictitious play, in JSFP, each player assumes that all other players make decisions jointly according to their joint empirical frequencies. Also see Section II for more details. In the case of JSFP with inertia, the convergence to a pure strategy Nash equilibrium is established for all generalized ordinal potential games [8].

The contributions of this paper can be summarized as follows. First of all, we present a new variant of fictitious play called average strategy fictitious play (ASFP) and prove its convergence for a congestion game under some reasonable assumptions. Note that the computational burden of each player in JSFP mainly comes from the utility updating process and increases with the number of possible actions. ASFP proposed in this paper reduces the computational burden of JSFP by getting rid of the above utility updating process. In ASFP, each player only obtains a weighted running average of all other players' actions and assumes that the total number of other players choosing one action is equal to the weighted running average corresponding to this action.

Secondly, we apply the results to the road pricing problem. The road pricing system is typically implemented for affecting motorists' route choices [9], [10] and trip timing choices [11], [12]. In particular, road pricing for socially beneficial trip timing is the motivating example of this paper. We formulate the trip timing problem as a congestion game. Congestion game as a special case of potential game was first introduced in [13], where a collection of homogeneous agents have to choose from a finite set of alternatives and the payoff of a player depends on the number of players choosing

each alternative. Different from [13], players' utilities are heterogeneous in this paper.

The remainder of the paper is organized as follows. Section II summarizes a series of notations to be used and some pertinent work to this paper. A repeated congestion game is formulated in Section III, where the structure of ASFP is also introduced. In Section IV, we first establish the existence of pure strategy Nash equilibria for the congestion game under investigation, and then show the convergence of ASFP with inertia to a pure strategy Nash equilibrium. In Section V, the results are applied in the design of road pricing scheme, and simulations based on the real traffic data in Singapore are also included. Finally, Section VI concludes the paper and discusses future work.

II. NOTATIONS AND RELATED WORK

Consider a repeated finite N-player game with the set of players $\mathcal{N} := \{1, 2, \ldots, N\}$ over consecutive stages $t = 0, 1, 2, \ldots$. For any $i \in \mathcal{N}$, $-i$ denotes the complementary set $\mathcal{N} \backslash \{i\}$. The action of player i at stage t is denoted by $x_i(t) \in X_i$, and X_i is the action set. For the rest of the paper, the argument t may be omitted when no confusion is caused. Let $x = (x_i, x_{-i})$ represent the action profile of all the players. The utility received by player i is denoted by $U_i(x_i, x_{-i})$ or simply $U_i(x)$.

The definition of potential game is given as follows [7]. Note that if a game admits a potential function, then all players tend to jointly optimize the potential function.

Definition 2.1: A finite N-player game with utility functions $U_1(x), U_2(x), \ldots, U_N(x)$ is called a *potential game* if there exists a potential function $\Phi(x)$ such that for every i,

$$U_i(x_i^1, x_{-i}) - U_i(x_i^2, x_{-i}) = \Phi(x_i^1, x_{-i}) - \Phi(x_i^2, x_{-i}) \quad (1)$$

for all possible $x_i^1 \in X_i, x_i^2 \in X_i, x_{-i} \in X_{-i}$.

An N-tuple of action variable x^* constitutes a (pure strategy) Nash equilibrium [2], if for all possible i, x_i,

$$U_i(x_i^*, x_{-i}^*) \geq U_i(x_i, x_{-i}^*). \quad (2)$$

It is well known that every potential game has at least one pure strategy Nash equilibrium, since a global maximum point of the potential function is obviously one Nash equilibrium [7].

Standard fictitious play is an iterative learning process incorporating the empirical frequencies of opponents' actions without assuming any information of other players' utilities [3]. Denote $f_i^{\hat{x}_i}(t) = \frac{1}{t} \sum_{s=0}^{t-1} I\{x_i(s) = \hat{x}_i\}$ as the empirical frequency of player i choosing action $\hat{x}_i \in X_i$ up to stage $t - 1$. Here $I\{\cdot\}$ is the indicator function. Every player assumes that other players make decisions randomly and independently according to those empirical frequencies, then the expected utility for player i choosing $\hat{x}_i \in X_i$ is

$$\bar{U}_i(\hat{x}_i, f_{-i}(t)) = \sum_{\hat{x}_{-i} \in X_{-i}} U(\hat{x}_i, \hat{x}_{-i}) \prod_{j \in -i} f_j^{\hat{x}_j}(t). \quad (3)$$

Let every player select an action that maximizes the expected utility (3) at every stage, then the empirical frequencies generated by fictitious play converge to a mixed strategy Nash equilibrium in all potential games [7]. However, fictitious play is computationally infeasible for large-scale games, since choosing an action for player i at every stage depends on a mapping over a joint space X_{-i}.

In JSFP, the empirical frequencies of the joint actions of other players, defined as $g_{-i}^{\hat{x}_{-i}}(t) = \frac{1}{t} \sum_{s=0}^{t-1} I\{x_{-i}(s) = \hat{x}_{-i}\}$, are tracked by player i. Every player assumes that other players make decisions randomly and jointly according to the above empirical frequencies. The expected utility for player i choosing $\hat{x}_i \in X_i$ becomes

$$\bar{U}_i(\hat{x}_i, g_{-i}(t)) = \sum_{\hat{x}_{-i} \in X_{-i}} U(\hat{x}_i, \hat{x}_{-i}) g_{-i}^{\hat{x}_{-i}}. \quad (4)$$

It turns out that the expected utility in JSFP can be calculated recursively as [8]

$$\begin{aligned} &\bar{U}_i(\hat{x}_i, g_{-i}(t+1)) \\ =&\frac{t}{t+1}\bar{U}_i(\hat{x}_i, g_{-i}(t)) + \frac{1}{t+1}U_i(\hat{x}_i, x_{-i}(t)). \end{aligned} \quad (5)$$

Thus, JSFP reduces the computational burden of standard fictitious play. The convergence of JSFP is also addressed for all generalized ordinal potential games in [8]. Next, we introduce ASFP that further reduces the computational burden of JSFP by getting rid of the utility updating process (5).

III. ASFP SETUP

We consider a congestion game where the action of player i at stage t is chosen from a finite collection of common resources $\mathcal{R} := \{r_1, r_2, \ldots, r_M\}$. For the ease of the presentation, we limit our attention to a single choice case, i.e., the cardinality of $x_i(t) \in \mathcal{R}$ is 1. However, the results presented here can be further extended to a multiple choices case. It is assumed that the utility received by player i can be divided into two parts as

$$U_i(x_i, x_{-i}) = V_{1i}(x_i) + V_2(n_{x_i}(x)), \quad (6)$$

where

$$n_{x_i}(x) = \sum_{j=1}^N I\{x_j = x_i\} \quad (7)$$

is the number of players choosing resource x_i. In (6), $V_{1i}(x_i)$ represents the fixed utility received by player i for using resource x_i, and $V_2(n_{x_i}(x))$ denotes the utility part due to the congestion of players using the same resource. Different from the congestion game introduced in [13], players' utilities are heterogeneous in (6).

At the initial stage $t = 0$, every player picks up an action arbitrarily from $\mathcal{R}$. Then at any stage $t \geq 1$, a system supervisor records $n_{r_l}(x(t-1))$ for all $l = 1, 2, \ldots, M$, and computes its weighted running average recursively as

$$\begin{aligned} &\bar{n}_{r_l}(0) = n_{r_l}(x(0)), \\ &\bar{n}_{r_l}(t) = (1 - \lambda)\bar{n}_{r_l}(t-1) + \lambda n_{r_l}(x(t-1)), \end{aligned} \quad (8)$$

where $\lambda \in (0, 1]$ is a constant weight on the newest information. In ASFP, the above weighted running average $\bar{n}_{r_l}(t)$ is broadcast by the system supervisor to every player at

the beginning of stage $t \geq 1$. For player $i \in \mathcal{N}$, it is easy to obtain the following quantity

$$\bar{n}_{r_l}^{-i}(t) = \bar{n}_{r_l}(t) - w_{r_l}^i(t), \quad (9)$$

where $w_{r_l}^i(0) = I\{x_i(0) = r_l\}$ and $w_{r_l}^i(t) = (1 - \lambda)w_{r_l}^i(t - 1) + \lambda I\{x_i(t - 1) = r_l\}$. Note that $\bar{n}_{r_l}^{-i}(t)$ represents the weighted running average of the number of players choosing resource r_l except for player i up to stage $t-1$. Furthermore, in ASFP, every player $i \in \mathcal{N}$ makes a presumption that the total number of other players choosing resource r_l at stage t is equal to the weighted running average $\bar{n}_{r_l}^{-i}(t)$. In this situation, the predicted utility for player i choosing $\hat{x}_i \in \mathcal{R}$ at stage t is given by

$$\bar{U}_i(\hat{x}_i, \bar{n}^{-i}(t)) = V_{1i}(\hat{x}_i) + V_2(\bar{n}_{\hat{x}_i}^{-i}(t) + 1), \quad (10)$$

and the set of actions as the best response of player i is defined as

$$\begin{aligned}&\mathrm{BR}_i(\bar{n}^{-i}(t))\\&:= \{\tilde{x}_i \in \mathcal{R} | \bar{U}_i(\tilde{x}_i, \bar{n}^{-i}(t)) = \max_{x_i \in \mathcal{R}} \bar{U}_i(x_i, \bar{n}^{-i}(t))\}.\end{aligned} \quad (11)$$

Remark 3.1: Different from JSFP, the utility updating process (5) is avoided in ASFP, and thus the burden for information processing of each player is reduced especially when the number of resources to be chosen from is large.

Next, given the best response set $\mathrm{BR}_i(\bar{n}^{-i}(t))$ for every player i at every stage $t \geq 1$, we will study convergence of ASFP.

IV. CONVERGENCE ANALYSIS OF ASFP

In this section, the convergence property of ASFP is established via two steps. First, we will prove the existence of a set of Nash Equilibria for the congestion game formulated in Section III. After that, we will show that players' actions generated by ASFP with some kind of inertia converge to one of Nash equilibria almost surely.

A. Existence of Pure Strategy Nash Equilibria

The following proposition shows that the congestion game defined in Section III is a potential game; thus, it inherits the desirable property of potential game regarding the existence of a pure strategy Nash equilibrium.

Proposition 4.1: A congestion game with utility functions given in (6) is a potential game with the potential function

$$\Phi(x) = \sum_{j=1}^{N} V_{1j}(x_j) + \sum_{l=1}^{M} \sum_{k=1}^{n_{r_l}(x)} V_2(k), \quad (12)$$

and it has at least one pure strategy Nash equilibrium.
Proof: For the case: $x_i^1 = x_i^2$, condition (1) obviously holds. For the case: $x_i^1 \neq x_i^2$, we have

$$\begin{aligned}&\Phi(x_i^1, x_{-i}) - \Phi(x_i^2, x_{-i})\\&= \sum_{l=1}^{M} \sum_{k=1}^{n_{r_l}(x_i^1, x_{-i})} V_2(k) - \sum_{l=1}^{M} \sum_{k=1}^{n_{r_l}(x_i^2, x_{-i})} V_2(k)\\&\quad + V_{1i}(x_i^1) - V_{1i}(x_i^2)\\&= V_2(n_{x_i^1}(x_i^1, x_{-i})) - V_2(n_{x_i^2}(x_i^2, x_{-i})) + V_{1i}(x_i^1) - V_{1i}(x_i^2)\\&= U_i(x_i^1, x_{-i}) - U_i(x_i^2, x_{-i}),\end{aligned}$$

i.e., condition (1) is satisfied. The proof is completed since every potential game has at least one pure strategy Nash equilibrium [7]. □

B. Convergence of Players' Learning Process

For the simplicity of the analysis, we make the following assumption on the utility induced by congestion. Its validity will be further justified in Section V on transportation modeling.

Assumption 1: The utility part induced by congestion in (6) is linear with respect to the number of players selecting the same resource, i.e.,

$$V_2(n_{x_i}(x)) = an_{x_i}(x) + b, \quad (13)$$

where a, b are constant parameters.

Following the idea in [8], [14], some sort of inertia, which plays an important role in the convergence analysis, is introduced in players' decision making process. In the presence of inertia, player $i \in \mathcal{N}$ stays with the previous action, i.e., $x_i(t) = x_i(t - 1)$, if there is no opportunity for utility improvement, i.e., $x_i(t-1) \in \mathrm{BR}_i(\bar{n}^{-i}(t))$; otherwise, he or she chooses an action from the set $\mathrm{BR}_i(\bar{n}^{-i}(t))$ with probability $\xi_i(t)$ and still stays with the previous action with probability $1 - \xi_i(t)$. The absorption property of Nash equilibria in ASFP with inertia is shown in the following proposition.

Proposition 4.2: Consider a congestion game with utility functions given in (6). Under Assumption 1, if the action profile $x(t)$ of all players generated by ASFP with inertia is a pure strategy Nash equilibrium at stage t, and $x_i(t) \in \mathrm{BR}_i(\bar{n}^{-i}(t))$ for every $i \in \mathcal{N}$, then $x(t + \tau) = x(t)$ for all $\tau > 0$.
Proof: First note that

$$n_{\hat{x}_i}(\hat{x}_i, x_{-i}(t)) = n_{\hat{x}_i}(x(t)) - I\{x_i(t) = \hat{x}_i\} + 1. \quad (14)$$

After substituting (8) into (9), we have

$$\begin{aligned}\bar{n}_{r_l}^{-i}(0) =& n_{r_l}(x(0)) - I\{x_i(0) = r_l\},\\\bar{n}_{r_l}^{-i}(t) =& (1 - \lambda)\bar{n}_{r_l}^{-i}(t - 1)\\&+ \lambda(n_{r_l}(x(t - 1)) - I\{x_i(t - 1) = r_l\}).\end{aligned} \quad (15)$$

It follows that

$$\begin{aligned}&\bar{U}_i(\hat{x}_i, \bar{n}^{-i}(t + 1))\\=& V_{1i}(\hat{x}_i) + V_2(\bar{n}_{\hat{x}_i}^{-i}(t + 1) + 1)\\=& a\{(1 - \lambda)\bar{n}_{r_l}^{-i}(t) + \lambda(n_{r_l}(x(t)) - I\{x_i(t) = r_l\})\}\\&+ a + b + V_{1i}(\hat{x}_i)\\=& (1 - \lambda)\bar{U}_i(\hat{x}_i, \bar{n}^{-i}(t)) + \lambda U_i(\hat{x}_i, x_{-i}(t)). \quad (16)\end{aligned}$$

Based on the condition $x_i(t) \in \mathrm{BR}_i(\bar{n}^{-i}(t))$, we have for all $\hat{x}_i$,

$$\bar{U}_i(x_i(t), \bar{n}^{-i}(t)) \geq \bar{U}_i(\hat{x}_i, \bar{n}^{-i}(t)). \quad (17)$$

In addition, $x(t)$ is a pure strategy Nash equilibrium, therefore for all $\hat{x}_i$,

$$U_i(x_i(t), x_{-i}(t)) \geq U_i(\hat{x}_i, x_{-i}(t)). \quad (18)$$

It follows from (16) that for all $\hat{x}_i$,

$$\bar{U}_i(x_i(t), \bar{n}^{-i}(t+1)) \geq \bar{U}_i(\hat{x}_i, \bar{n}^{-i}(t+1)), \quad (19)$$

i.e., $x_i(t) \in \mathrm{BR}_i(\bar{n}^{-i}(t+1))$. Then we can obtain that $x(t+1) = x(t)$ based on players' inertia. The rest of the proof follows by induction. $\square$

Two technical assumptions are essential to convergence analysis of ASFP: one on players' willingness to optimize the predicted utility, and the other on players' difference between any two distinct strategies.

Assumption 2: For every player $i \in \mathcal{N}$ and for every stage $t \geq 1$, there exist constants δ_1 and δ_2 such that

$$0 < \delta_1 \leq \xi_i(t) \leq \delta_2 < 1. \quad (20)$$

Assumption 3: For every player $i \in \mathcal{N}$,

$$U_i(x_i^1, x_{-i}) \neq U_i(x_i^2, x_{-i}) \quad (21)$$

for all possible x_i^1, x_i^2, x_{-i}.

As the main result of this paper, the convergence property of ASFP is provided in the next theorem.

Theorem 4.1: Consider a congestion game with utility functions given in (6). Under Assumptions 1, 2 and 3, the action profile of all players generated by ASFP with inertia is convergent to a pure strategy Nash equilibrium.

Proof: According to (16), if $x(t)$ is repeated T times, i.e., $x(t) = x(t+1) = \cdots = x(t+T-1)$, then

$$\bar{U}_i(\hat{x}_i, \bar{n}^{-i}(t+T-1)) = (1-\lambda)^{T-1} \bar{U}_i(\hat{x}_i, \bar{n}^{-i}(t)) + (1-(1-\lambda)^{T-1}) U_i(\hat{x}_i, x_{-i}(t)).$$

For $\lambda \in (0, 1]$, there exists a sufficiently large T independent of t such that $\mathrm{BR}_i(\bar{n}^{-i}(t+T-1)) = \mathrm{BR}_i(x_{-i}(t))$, where $\mathrm{BR}_i(x_{-i}(t))$ is defined as the best response of player i with respect to the action profile of other players $x_{-i}(t)$, i.e.,

$$\mathrm{BR}_i(x_{-i}(t)) := \{\tilde{x}_i \in \mathcal{R} | U_i(\tilde{x}_i, x_{-i}(t)) = \max_{x_i \in \mathcal{R}} U_i(x_i, x_{-i}(t))\}. \quad (22)$$

Note that the probability of the above event is at least $(1-\delta_2)^{N(T-1)}$, and under Assumption 3 the cardinality of $\mathrm{BR}_i(x_{-i}(t))$ is 1. If $x(t)$ is a pure strategy Nash equilibrium, then the proof is completed based on Proposition 4.2. Otherwise, there exists at least one player j such that $U_j(x_j(t), x_{-j}(t)) < U_j(\hat{x}_j, x_{-j}(t))$. If player j switches its action to $\hat{x}_j$ and all other players stay with their previous actions, i.e., $x(t+T) = (\hat{x}_j, x_{-j}(t))$, and $x(t+T)$ is repeated T times, then $\mathrm{BR}_i(\bar{n}^{-i}(t+2T-1)) = \mathrm{BR}_i(x_{-i}(t+T))$ for a sufficiently large T. The corresponding probability is at least $\delta_1(1-\delta_2)^{NT-1}$. According to Proposition 4.1, we have $\Phi(x(t)) < \Phi(x(t+T))$. Again, if $x(t+T)$ is a pure strategy Nash equilibrium, we are done. Otherwise, we can continue the above argument and construct a sequence of action profiles as $x(t), x(t+T), \ldots, x(t+KT)$ such that

$$\Phi(x(t)) < \Phi(x(t+T)) < \cdots < \Phi(x(t+KT)). \quad (23)$$

Note that the cardinality of the decision space of all players is M^N, which is finite. In addition, there always exists at least one pure strategy Nash equilibrium for the underlying congestion game. Therefore, there exists a finite $K < M^N$ such that $x(t+KT)$ is a pure strategy Nash equilibrium. In summary, we can conclude that $x(t+KT)$ is a pure strategy Nash equilibrium with $K < M^N$ and with a positive probability at least $(1-\delta_2)^{N(T-1)}\{\delta_1(1-\delta_2)^{NT-1}\}^K$, which implies that $x(t)$ generated by ASFP with inertia converges to a pure strategy Nash equilibrium almost surely. $\square$

Remark 4.1: As we can see from the proof of Theorem 4.1, if there are multiple Nash equilibria for the underlying game, then the action profile may converge to a pure strategy Nash equilibrium corresponding to only a local maximum point of the potential function. Note that this kind of phenomenon also motivates research work on inefficiency of equilibria [15], which can be one of our future directions.

V. APPLICATION OF ASFP TO ROAD PRICING

Consider the case where a group of people travel along one road during a given period (e.g., 7:30am-9:30am) on a daily basis.

A. Trip Timing without Road Price

For the problem of trip timing, a collection of time intervals is considered as the set of resources to be chosen by every road user. Without road price, each driver decides his or her departure time by taking the average travel speed and his or her own preferred departure time into account. Suppose that $n_{r_i}(x)$ is the number of vehicles on the road choosing departure time interval r_i and the length of the road is L. A common assumption in the area of transportation theory is that the average travel speed is a linear function of traffic density, i.e., $n_{r_i}(x)/L$; see, e.g., [16], [17]. In this case, we set the utility parts V_{1i}, V_2 in (6) as

$$V_{1i}(x_i) = \alpha_i |x_i - \hat{T}_i|, \quad (24)$$
$$V_2(n_{x_i}(x)) = a n_{x_i}(x) + b, \quad (25)$$

where $\hat{T}_i$ is the preferred departure time of driver i, and α_i, a, b are constant parameters. Note that $|x_i - \hat{T}_i|$ is the time difference between actual and preferred departure time, and $\alpha_i \leq 0$ may be different for different road user. In addition, since the average travel speed is always decreasing with respect to the number of vehicles on the road, we have $a < 0$.

In general, the road manager (i.e., the system supervisor) monitors the traffic flows in the traffic system, and thus the weighted running average defined in (8) can be computed by the road manager. The weighted running average actually describes the crowdedness of the road during each time interval based on the historical traffic situation. Suppose that the road manager broadcasts (8) to every driver. Then, the above trip timing problem fits within the congestion game formulated in Section III. Moreover, if every driver generates his or her action based on ASFP with inertia, then the convergence of the action profile of all drivers is ensured under the conditions of Theorem 4.1.

B. Trip Timing under Dynamic Road Pricing

Assume that the road price determined by the road manager is applied to achieve some kind of social goal and it is charged when the driver enters the road. We consider the case where the road price is a function of the number of vehicles on the road and the utility function (6) is modified into

$$U_i(x_i, x_{-i}) = V_{1i}(x_i) + V_2(n_{x_i}(x)) + cp(n_{x_i}(x)), \quad (26)$$

where $p(n_{x_i}(x))$ is the road price to be designed, and $c < 0$ is an additional parameter. We can derive the following result.

Theorem 5.1: If the road price is set according to

$$p(k) = \frac{a}{c}(k - 1), \quad (27)$$

then the congestion game with utility functions given in (26) is a potential game with the potential function

$$\tilde{\Phi}(x) = \sum_{j=1}^{N} \{V_{1j}(x_j) + V_2(n_{x_j}(x))\}. \quad (28)$$

Proof: For $\tilde{\Phi}(x)$ in (28) and $x_i^1 \neq x_i^2$, we have

$$
\begin{aligned}
&\tilde{\Phi}(x_i^1, x_{-i}) - \tilde{\Phi}(x_i^2, x_{-i}) \\
=&V_{1i}(x_i^1) + n_{x_i^1}(x_i^1, x_{-i})[an_{x_i^1}(x_i^1, x_{-i}) + b] \\
&+ n_{x_i^2}(x_i^1, x_{-i})[an_{x_i^2}(x_i^1, x_{-i}) + b] \\
&- V_{1i}(x_i^2) - [n_{x_i^2}(x_i^1, x_{-i}) + 1][an_{x_i^2}(x_i^1, x_{-i}) + a + b] \\
&- [n_{x_i^1}(x_i^1, x_{-i}) - 1][an_{x_i^1}(x_i^1, x_{-i}) - a + b] \\
=&V_{1i}(x_i^1) - V_{1i}(x_i^2) + 2a[n_{x_i^1}(x_i^1, x_{-i}) - n_{x_i^2}(x_i^1, x_{-i}) - 1].
\end{aligned}
$$

Note that

$$n_{x_i^2}(x_i^2, x_{-i}) = n_{x_i^2}(x_i^1, x_{-i}) + 1. \quad (29)$$

Based on (26) and (27), we can obtain that for $x_i^1 \neq x_i^2$,

$$
\begin{aligned}
&U_i(x_i^1, x_{-i}) - U_i(x_i^2, x_{-i}) \\
=&V_{1i}(x_i^1) + an_{x_i^1}(x_i^1, x_{-i}) + b + cp(n_{x_i^1}(x_i^1, x_{-i})) \\
&- V_{1i}(x_i^2) - an_{x_i^2}(x_i^2, x_{-i}) - b - cp(n_{x_i^2}(x_i^2, x_{-i})) \\
=&V_{1i}(x_i^1) - V_{1i}(x_i^2) + 2a[n_{x_i^1}(x_i^1, x_{-i}) - n_{x_i^2}(x_i^1, x_{-i}) - 1] \\
=&\tilde{\Phi}(x_i^1, x_{-i}) - \tilde{\Phi}(x_i^2, x_{-i}).
\end{aligned}
$$

The rest of the proof follows similarly to Proposition 4.1. $\square$

Note that the pricing scheme given in (27) is dynamic. In addition, $\tilde{\Phi}(x)$ in (28) represents the overall utility of all road users in the absence of road price, and thus $\tilde{\Phi}(x)$ can be considered as a social welfare to be maximized by the system supervisor. According to Remark 4.1, the action profile generated by ASFP with inertia may converge to a local maximum point of $\tilde{\Phi}$. However, starting from the same set of initial conditions, the proposed pricing strategy can always improve the overall utility compared to the case without road pricing; see next subsection for a case study of Singapore.
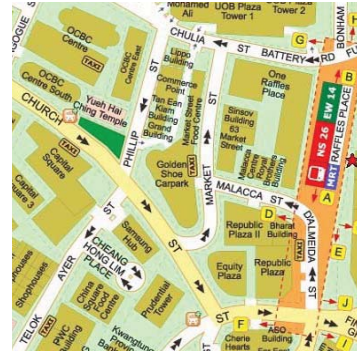

Fig. 1. The map of Church street, Singapore.

C. Case Study of Singapore

We take Church street, Singapore (Fig. 1) as an example. Land Transport Authority (LTA) of Singapore and the Comfort Taxi company provide us with loop counts and comfort taxi data for the month of August 2010, respectively. The loop counts record the number of vehicles passing through all inductive loop detectors, and the comfort taxi data contain the information on the changing of Taxies' GPS location with respect to time. The average travel speed can be computed based on the comfort taxi data, and the traffic flow is given by the loop counts. Then, the traffic density can be calculated as the ratio of traffic flow to average travel speed. The relationship between average travel speed and traffic density for Church street is plotted in Fig. 2, which is approximately linear. The length of Church street is about 0.3
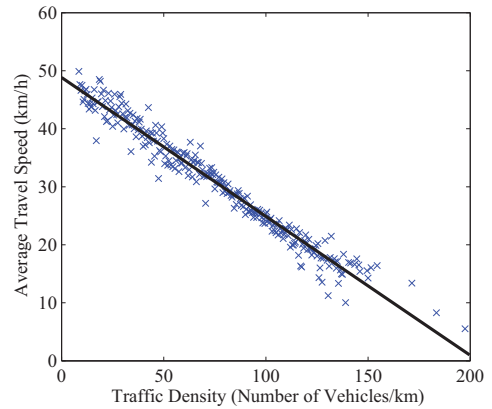

Fig. 2. The relationship between average travel speed and traffic density for Church street, Singapore.

kilometer, then we can derive $a = -0.798$ km/h/vehicle, $b = 48.835$ km/h in (25).

Suppose that there are 200 road users using Church street from 7:30am to 9:30am everyday. In the simulation, the value of α_i, $i = 1, 2, \ldots, 500$, is randomly generated according to a uniform distribution within the interval $[-150, -50]$, while $\hat{T}_i, i = 1, 2, \ldots, 500$, is randomly generated according to a triangular distribution within the interval [7:30am, 9:30am] and with a peak at 8am. In practice, the distribution of α_i and $\hat{T}_i$ may be determined via household survey, which is out of

the scope of this paper. We divide the time horizon [7:30am, 9:30am] into 8 non-overlapping intervals, i.e., $M = 8$. Each interval has a length of 0.25 hour.

We first consider the case without road price. As we can see from Fig. 3, the action profile of all players generated by ASFP with inertia is convergent, and we can further check that the convergent point is a pure strategy Nash equilibrium. For the case with pricing scheme in (27), the action profile of
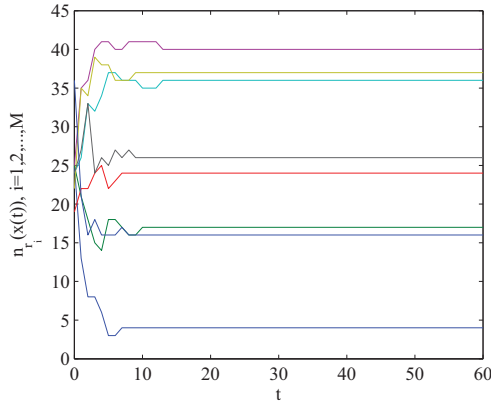

Fig. 3. Number of vehicles in each departure time interval without road price over day $t = 0, 1, 2, \ldots$.

all players generated by ASFP with inertia is still convergent to a pure strategy Nash equilibrium as shown in Fig. 4. Starting from the same set of initial conditions, the overall utility for the case without road price is $\tilde{\Phi}(x) = 3565.2$ and for the case with pricing scheme (27) is $\tilde{\Phi}(x) = 3642.9$. It can be observed that the proposed pricing strategy improves the overall utility of all players. By comparing Fig. 3 with Fig. 4, we can see that the proposed road pricing scheme actually shifts a portion of people from relatively more congested time intervals to less congested ones.
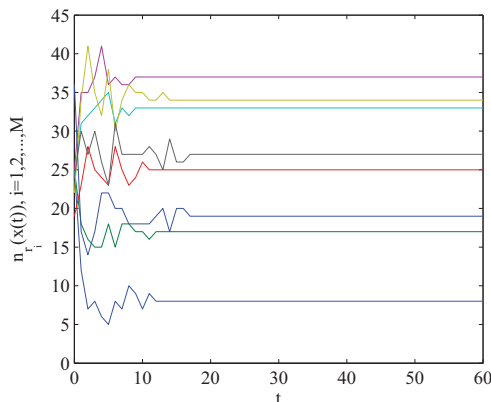

Fig. 4. Number of vehicles in each departure time interval under pricing scheme (27) over day $t = 0, 1, 2, \ldots$.

VI. CONCLUSIONS

In this paper, the so-called average strategy fictitious play has been introduced for large-scale repeated congestion games. It avoids the utility updating process in joint strategy fictitious play studied in [8]. The convergence property of ASFP with inertia has been established, and the results have been applied in road pricing design to achieve socially beneficial trip timing. Note that a generalization of the results in this paper to the case without broadcasting of system supervisor is considered in [18] by using distributed consensus. Future work includes the relaxation of linearity assumption on the utility induced by congestion, the conditions for the uniqueness of Nash equilibrium, and other forms of broadcasted information, e.g., coordinated actions/signals.

VII. ACKNOWLEDGMENTS

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